Math 579 Fall 2013 Exam 1 Solutions

1. 55 distinct integers are selected, all from $[1, 100]$. Prove that some pair differs by 12.

   Consider the following 52 pigeonholes: $(1, 13)$, $(2, 14)$, $(3, 15)$, $(4, 16)$, $(5, 17)$, $(6, 18)$, $(7, 19)$, $(8, 20)$, $(9, 21)$, $(10, 22)$, $(11, 23)$, $(12, 24)$, $(25, 37)$, $(26, 38)$, $(27, 39)$, $(28, 40)$, $(29, 41)$, $(30, 42)$, $(31, 43)$, $(32, 44)$, $(33, 45)$, $(34, 46)$, $(35, 47)$, $(36, 48)$, $(49, 61)$, $(50, 62)$, $(51, 63)$, $(52, 64)$, $(53, 65)$, $(54, 66)$, $(55, 67)$, $(56, 68)$, $(57, 69)$, $(58, 70)$, $(59, 71)$, $(60, 72)$, $(73, 85)$, $(74, 86)$, $(75, 87)$, $(76, 88)$, $(77, 89)$, $(78, 90)$, $(79, 91)$, $(80, 92)$, $(81, 93)$, $(82, 94)$, $(83, 95)$, $(84, 96)$, (97), (98), (99), (100). Each number from 1 to 100 appears exactly once, so there is a natural mapping of our 55 numbers into here. Since $55 > 52$, by PHP there must be a collision; but the pigeonholes were chosen so that if two numbers are chosen from the same one, they differ by 12.

2. 300 points are placed within a unit cube. Prove that you can choose some 12 of these, all within 0.6 of each other.

   Divide the cube into $3 \times 3 \times 3 = 27$ cubes, as a Rubik's cube. By PHP, one of these small cubes must contain at least $\lceil \frac{300}{27} \rceil = 12$ points. This small cube has side length $1/3$, hence diagonal length $\sqrt{(1/3)^2 + (1/3)^2 + (1/3)^2} \approx 0.577$; hence any pair of points within are at less than 0.6 apart.

3. Use the PHP to prove that there is some $n \in \mathbb{N}$ such that $44^n - 1$ is divisible by 13.

   Consider $s_n = 44^n$. Divide each of $s_1, s_2, \ldots, s_{14}$ by 13. Since only 13 remainders are possible, there are i, j with $1 \le i < j \le 14$ with s_i, s_j having the same remainder. We calculate $s_j - s_i = 13(q_j - q_i)$, so $13|(44^j - 44^i)$. Hence, $13|44^i(44^{j-i} - 1)$. Since $\gcd(13, 44) = 1$, in fact $13|(44^{j-i} - 1)$, as desired.

4. 17 distinct integers are selected, all from $[1, 33]$. Prove that some pair among these has greatest common divisor 1.

   This is just a small tweak of problem 30. Consider the following 16 pigeonholes: $(1, 2)$, $(3, 4)$, $(5, 6)$, $(7, 8)$, $(9, 10)$, $(11, 12)$, $(13, 14)$, $(15, 16)$, $(17, 18)$, $(19, 20)$, $(21, 22)$, $(23, 24)$, $(25, 26)$, $(27, 28)$, $(29, 30)$, $(31, 32, 33)$. Each number from 1 to 33 appears exactly once, so there is a natural mapping of our 17 numbers into here. Since $17 > 16$, by PHP there must be a collision; but the pigeonholes were chosen so that any two numbers from the same one have gcd 1.

   Alternate tweak: Start with $(1, 2, 3)$, then $(4, 5), (6, 7), \ldots, (32, 33)$.

5. Prove that there is a positive integer n such that the distance from $n\pi$ to the nearest integer is less than 10^{-100}.

   Recall the following lemma, proved in class:

   Lemma: Let $\alpha \in \mathbb{R}$, $Q \in \mathbb{N}$. Then there are $p, q \in \mathbb{N}$ satisfying $0 < q \le Q$ and $|\alpha - \frac{p}{q}| < \frac{1}{Qq}$.

   Apply this lemma with $\alpha = \pi, Q = 10^{100}$; it gives you p, q so that $|\pi - \frac{p}{q}| < \frac{1}{10^{100} q}$. Multiply both sides by q to get $|\pi q - p| < 10^{-100}$, so we may take $n = q$.